Homework 8: Solutions

5.4.2: Change order and evaluate:

$$\int_0^1 \int_y^1 \sin(x^2)\, dx\, dy$$

The region we're integrating over is the triangle with end-points $(0,0), (1,0)$ and $(1,1)$. Thus, changing the order yields:

$$\int_0^1 \int_0^x \sin(x^2)\, dy\, dx$$

Which can be evaluated to:

$$\begin{aligned}
\int_0^1 \int_0^x \sin(x^2)\, dy\, dx &= \int_0^1 [y \sin(x^2)]_0^x\, dx \\
&= \int_0^1 x \sin(x^2)\, dx \\
&= \frac{1}{2} \int_0^1 \sin(x^2)\, 2x dx \\
&= \frac{1}{2} \int_0^1 \sin(u)\, du \\
&= \frac{1}{2}(-\cos(1) + \cos(0)) = \frac{1 - \cos(1)}{2}
\end{aligned}$$

5.4.5: Change order and evaluate:

$$\int_0^1 \int_{\sqrt{y}}^1 e^{x^3}\, dx\, dy$$

x ranges from $x = \sqrt{y}$ to $x = 1$. Notice $x = \sqrt{y}$ is the same as saying $y = x^2$ and $x \geq 0$. Thus the region we're integrating over is bounded by the x-axis, the parabola $y = x^2$ and the line $x = 1$. Thus integrating in the y direction first, we would have to go from 0 to x^2.

$$\int_0^1\int_{\sqrt{y}}^1 e^{x^3}\,dx\,dy = \int_0^1\int_0^{x^2} e^{x^3}\,dy\,dx$$
$$= \int_0^1 e^{x^3}x^2\,dx$$
$$= \frac{1}{3}\int_0^1 e^{x^3}3x^2dx$$
$$= \frac{1}{3}\int_0^1 e^u\,du$$
$$= \frac{1}{3}(e-1)$$

5.4.8: Show:

$$\frac{1}{2}(1-\cos(1)) \le \int_0^1\int_0^1 \frac{\sin(x)}{1+(xy)^4}\,dx\,dy \le 1.$$

Since x and y are both between 0 and 1, the denominator, $1+(xy)^4$, is at least 1 and at most 2. i.e.

$$\frac{1}{2} \le \frac{1}{1+(xy)^4} \le \frac{1}{1}.$$

Also, since $\sin(x)$ is positive for x in the interval $[0,1]$, we can multiple all sides by $\sin(x)$. Then notice that $\sin(x) \le 1$ to obtain:

$$\frac{\sin(x)}{2} \le \frac{\sin(x)}{1+(xy)^4} \le \sin(x) \le 1.$$

Finally, integrating over the square $[0,1]\times[0,1]$ yields:

$$\frac{1}{2}\int_0^1\int_0^1 \sin(x)\,dx\,dy \le \int_0^1\int_0^1 \frac{\sin(x)}{1+(xy)^4}\,dx\,dy \le \int_0^1\int_0^1 dx\,dy.$$

The left hand side evaluates to

$$\frac{1}{2}\int_0^1\int_0^1 \sin(x)\,dx\,dy = \frac{1}{2}\int_0^1 (-\cos(1))-(-\cos(0))\,dy = \frac{1}{2}(1-\cos(1)),$$

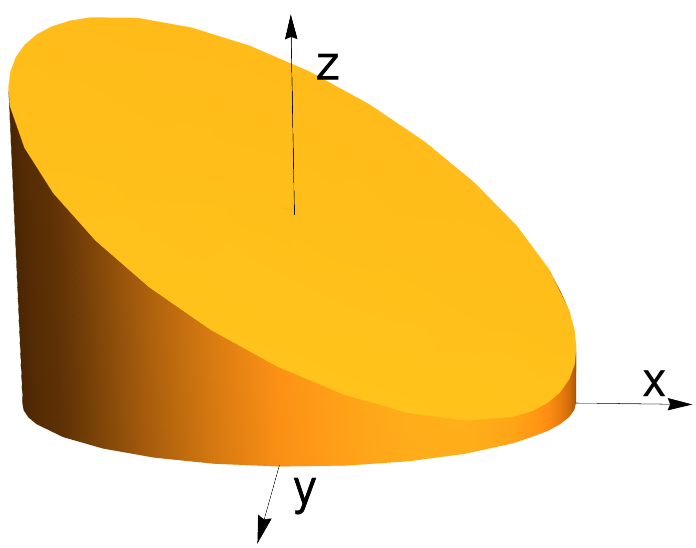
giving us:

$$\frac{1}{2}(1 - \cos(1)) \leq \int_0^1 \int_0^1 \frac{\sin(x)}{1 + (xy)^4} \, dx \, dy \leq 1.$$

5.5.12: Find the volume of the solid bounded by

$$x^2 + 2y^2 = 2, \; z = 0, \; \text{and } x + y + 2z = 2$$

$x^2 + 2y^2 = 2$ defines a vertical cylinder that crosses the xy-plane in an ellipse. $z = 0$ is the xy-plane. Since the ellipse in the xy-plane given by $x^2 + 2y^2 = 2$ does not intersect the line $x + y = 2$, the plane $x + y + 2z = 2$ crosses the cylinder above the xy-plane. Thus the region looks like:



For fixed (x, y), the values of z range between the planes $z = 0$ and $x + y + 2z = 2$, (so 0 to $(2 - x - y)/2$). For a fixed y, since $x^2 + 2y^2 = 2$, we get that x ranges between, $-\sqrt{2-2y^2}$ to $\sqrt{2-2y^2}$. Thus we get the volume V is given by:

$$
\begin{aligned}
V &= \int_{-1}^{1}\int_{-\sqrt{2-2y^2}}^{\sqrt{2-2y^2}}\int_{0}^{1-\frac{x+y}{2}} dz\, dx\, dy \\
&= \int_{-1}^{1}\int_{-\sqrt{2-2y^2}}^{\sqrt{2-2y^2}} [(1 - \frac{y}{2}) - \frac{x}{2}]\, dx\, dy \\
&= \int_{-1}^{1} 2(1 - \frac{y}{2})\sqrt{2-2y^2} - 0\, dy \\
&= \sqrt{2}\int_{-1}^{1} 2\sqrt{1-y^2} - y\sqrt{1-y^2}\, dy \\
&= 2\sqrt{2}\int_{-1}^{1} \sqrt{1-y^2}\, dy - \sqrt{2}\int_{-1}^{1} y\sqrt{1-y^2}\, dy \\
&= \sqrt{2}\pi - 0 = \sqrt{2}\pi.
\end{aligned}
$$

Where in the second last line, the first integral we identify as the area of a semi-circle of radius 1, and the second integral is 0, since it is the integral of an odd function over a symmetric interval.

5.5.22: Evaluate

$$\iiint_W (x^2 + y^2)\, dx\, dy\, dz$$

where W is the pyramid with top vertex $(0, 0, 1)$ and base vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, and $(1, 1, 0)$.

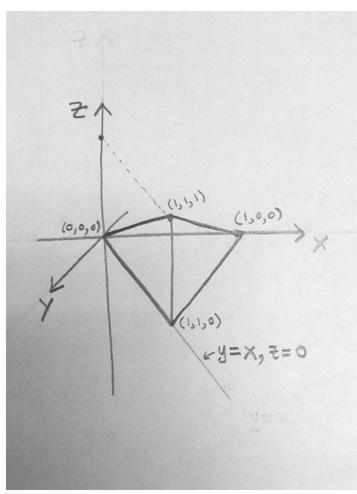
The horizontal cross-section of this pyramid at height z is a square with one corner at $(0, 0, z)$ and side length $1 - z$. Thus both x and y range from 0 to $1 - z$, and the integral becomes:

$$
\begin{aligned}
\iiint_W (x^2+y^2)\,dx\,dy\,dz &= \int_0^1 \int_0^{1-z} \int_0^{1-z} (x^2+y^2)\,dx\,dy\,dz \\
&= \int_0^1 \int_0^{1-z} \left(\frac{(1-z)^3}{3} + (1-z)y^2\right) dy\,dz \\
&= \int_0^1 \left(\frac{(1-z)^4}{3} + \frac{(1-z)^4}{3}\right) dz \\
&= \frac{2}{3} \int_0^1 (1-z)^4\,dz \\
&= \frac{2}{3} \left[\frac{(1-z)^5}{-5}\right]_0^1 \\
&= \frac{2}{3} \left(0 - \frac{1}{-5}\right) = \frac{2}{15}.
\end{aligned}
$$

5.5.24: (a) Sketch the region for

$$\int_0^1 \int_0^x \int_0^y f(x,y,z)\,dz\,dy\,dx.$$

The region is a tetrahedron bounded by the 4 planes $z = 0, z = y, y = x$ and $x = 1$. In other words, it is the tetrahedron with its four vertices the points $(0,0,0), (1,0,0), (1,1,0)$ and $(1,1,1)$:



(b) Change order to $dx\,dy\,dz$.

x ranges between the planes $x = y$ and $x = 1$. After x is integrated, we're left with an integral $dy\,dz$. We want the region in the yz-plane that we're integrating over. This is the projection of the tetrahedron to the yz-plane, and we see it is a triangle with vertices $(0,0,0),(0,1,0)$ and $(0,1,1)$. Thus we need to integrate y from $y = z$ to $y = 1$. z then ranges from 0 to 1. Thus we get that the integral is:

$$\int_0^1\int_z^1\int_y^1 f(x,y,z)\,dx\,dy\,dz.$$

6.1.2: Determine if the functions $T:\mathbb{R}^3 \to \mathbb{R}^3$ are one-to-one and/or onto:

$$(a) \quad T(x,y,z) = (2x + y + 3z, 3y - 4z, 5x)$$

First note that if we let A be the matrix

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 0 & 3 & -4 \\ 5 & 0 & 0 \end{pmatrix},$$

we have (if we think of (x,y,z) as a column vector)

$$T(x,y,z) = A(x,y,z).$$

The matrix A is invertible, [since its determinant is $5 \times (-4 - 9) \neq 0$], so for any vector (a,b,c), we can solve $T(x,y,z) = (a,b,c)$ by simply multiplying both sides by A^{-1} to get $(x,y,z) = A^{-1}(a,b,c)$

Thus for every (a,b,c) we can find an (x,y,z) that is sent by T to it (hence onto), and we can only find one such (x,y,z) (hence one-to-one). So the function is both onto and one-to-one.

$$(b) \quad T(x,y,z) = (y\sin(x), z\cos(y), xy)$$

Both $(0,0,0)$ and $(1,0,0)$ are sent to the point $(0,0,0)$, so the function is not one-to-one.

Also, the point $(0,0,1)$ is not in the image, since for the last coordinate to be 1, we must have $xy = 1$ so both x and y are ± 1. But this forces the first coordinate to be $\sin(1)$ which is not equal to 0. Thus the function is not onto either.

$$(c) \quad T(x,y,z) = (xy, yz, xz)$$

Both $(1,1,1)$ and $(-1,-1,-1)$ are sent to the point $(1,1,1)$, so the function is not one-to-one.

As for onto, nothing is sent to $(1,1,0)$, since $xz = 0$ implies either $x = 0$ or $z = 0$. Either way, one of the first two coordinates is 0. So the function is not onto either.

$$(d) \quad T(x,y,z) = (e^x, e^y, e^z)$$

This function cannot be onto since e is positive, and the power of a positive number is always (strictly) positive. Thus, for instance, nothing is sent to $(0,0,0)$ or to $(-1,-3,-11)$.

The function is one-to-one though, as can be seen as follows. Suppose, two points get sent to the same value, i.e. $T(x,y,z) = T(x',y',z')$. Then $e^x = e^{x'}$, and taking logarithms on both sides yields, $x = x'$ as the only real solution. Similarly $y = y'$ and $z = z'$. Thus the points (x,y,z) and (x',y',z') are actually the same. Since this works in general, it proves the claim. Thus T is one-to-one but not onto.

6.1.10: Find a T that sends the parallelogram D^* with vertices $(-1,3), (0,0), (2,-1), (1,2)$ to the square $D = [0,1] \times [0,1]$.

Linear maps (those given by a matrix) send squares to parallelograms, so we can try to find a linear map. The matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ sends $(1,0)$ to its first column, and $(0,1)$ to its second column (as you may easily check).

Thus the matrix $A = \begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix}$ sends $(1,0)$ to $(2,-1)$ and sends $(0,1)$ to $(-1,3)$ (and hence sends D to D^*). Since we want a map from D^* to D, we are looking for the inverse matrix,

$$A^{-1} = \frac{1}{5}\begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}.$$

Thus T is given by,

$$T(x,y) = A^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{5}(3x+y, x+2y).$$

6.2.1: Suggest substitution and find its Jacobian.

$$(a) \qquad \iint_R (3x+2y)\sin(x-y)\, dA$$

Letting $u = 3x + 2y$ and $v = x - y$ would greatly simplify the integral. The resulting integral would become

$$\iint_{R'} u\sin(v)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dA$$

where $\left|\frac{\partial(x,y)}{\partial(u,v)}\right|$ is the absolute value of the determinant of the matrix

$$\begin{pmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{pmatrix}$$

is just a number in this case.

Moreover, the transformation $(u,v) = T(x,y) = (3x+2y, x-y)$ is one-to-one and onto, as it must be in order to be a valid change of coordinates. So it has an inverse T^{-1} such that $T \circ T^{-1} = \text{Identity}$. Using chain rule, we thus get,

$$\frac{\partial(x,y)}{\partial(u,v)} \frac{\partial(u,v)}{\partial(x,y)} = 1.$$

We do this since, $\left|\frac{\partial(u,v)}{\partial(x,y)}\right|$ is much easier to compute, (you may also solve for (x,y) in terms of (u,v) and compute the Jacobian directly).

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 3 & 2 \\ 1 & -1 \end{vmatrix} = -5.$$

Thus

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{-5}$$

and the Jacobian is its absolute value $\frac{1}{5}$.

$$(b) \quad \iint_R e^{(-4x+7y)} \cos(7x - 2y)\, dA$$

Let $u = -4x + 7y$ and $v = 7x - 2y$. Then

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} -4 & 7 \\ 7 & -2 \end{vmatrix} = 8 - 49 = -41.$$

Thus

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{-41}$$

and the Jacobian is its absolute value 1/41.

6.2.2: Suggest substitution and find its Jacobian.

$$(a) \quad \iint_R (5x + y)^3 (x + 9y)^4\, dA$$

Let $u = 5x + y$ and $v = x + 9y$. Then

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 5 & 1 \\ 1 & 9 \end{vmatrix} = 45 - 1 = 44.$$

Thus

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{44}$$

is the Jacobian.

$$(b) \qquad \iint_R x\sin(6x+7y) - 3y\sin(6x+7y)\,dA$$

Note that the integral may be written as $\iint_R (x-3y)\sin(6x+7y)\,dA$, so let $u = x-3y$ and $v = 6x+7y$. Then

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & -3 \\ 6 & 7 \end{vmatrix} = 7 + 18 = 25.$$

Thus

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{25}$$

is the Jacobian.